



CURVATURE IN HILBERT GEOMETRIES

ÁRPÁD KURUSA

ABSTRACT. We provide more transparent proofs for the facts that the curvature of a Hilbert geometry in the sense of Busemann can not be non-negative and a point of non-positive curvature is a projective center of the Hilbert geometry. Then we prove that the Hilbert geometry has non-positive curvature at its projective centers, and that a Hilbert geometry is a Cayley–Klein model of Bolyai’s hyperbolic geometry if and only if it has non-positive curvature at every point of its intersection with a hyperplane. Moreover a 2-dimensional Hilbert geometry is a Cayley–Klein model of Bolyai’s hyperbolic geometry if and only if it has two points of non-positive curvature and its boundary is twice differentiable where it is intersected by the line joining those points of non-positive curvature.

1. INTRODUCTION

A *Hilbert geometry* is a pair $(\mathcal{I}, d_{\mathcal{I}})$ of an open, strictly convex domain $\mathcal{I} \subset \mathbb{R}^n$, and the *Hilbert metric* [2, page 297] $d_{\mathcal{I}}: \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}$ given by

$$(1.1) \quad d_{\mathcal{I}}(X, Y) = \begin{cases} 0, & \text{if } X = Y, \\ \frac{1}{2} |\ln(A, B; X, Y)|, & \text{if } X \neq Y, \text{ where } \overline{AB} = \mathcal{I} \cap XY. \end{cases}$$

Every geodesic $\tilde{\ell}$ of a Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ is the intersection $\mathcal{I} \cap \ell$ of \mathcal{I} with a straight line ℓ .

Busemann posed the problem [3, 34th on p. 406] if a Hilbert geometry that has non-positive curvature at every point is a Cayley–Klein model of Bolyai’s hyperbolic geometry. This was affirmatively answered in [4, Theorem, p. 119], where Kelly and Strauss showed that if a point in a Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ has non-positive curvature then it is a projective center of \mathcal{I} . They finished [4] by a conjecture that a Hilbert geometry can contain no points of non-negative curvature. This was proved in [6], where Kelly and Strauss closed the paper by discussing the problem if

(1.2) *a projective center has non-positive curvature.*

In this paper we provide a bit more transparent proofs for the above mentioned results of Kelly and Strauss, and then we prove (1.2) in Theorem 4.2. Finally we obtain sharper affirmative answers for Busemann’s problem [3, 34th on p. 406] in Section 5 as easy consequences.

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2. NOTATIONS AND PRELIMINARIES

Points of \mathbb{R}^n are denoted as A, B, \dots . The open segment with endpoints A and B is denoted by \overline{AB} , and AB denotes the line through A and B .

We denote the *affine ratio* of the collinear points A, B and C by $(A, B; C)$ that satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$. The *affine cross ratio* of the collinear points A, B and C, D is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [2, page 243].

In this article \mathcal{I} is an open, strictly convex domain in \mathbb{R}^n , where $n \geq 2$. We shall use without further notice the well-known fact [8, Theorem 25.3], that a convex function has both one-sided derivative at every point, and its derivative is strictly monotone, hence it is differentiable everywhere except at most a countable set. Moreover, a convex function has a second-order quadratic expansion at almost every point of its domain by Alexandrov's theorem [1] (see [9, Theorem 2.1]). These are called *Alexandrov points*, and in the expansions the usual big- O notation is used.

Given a point $P \in \mathcal{I}$, the *polar* P^* of P is defined as the locus of every point X that is the harmonic conjugate of P with respect to A and B , where $\overline{AB} = \mathcal{I} \cap PX$. It is easy to see [7, p. 64] that the polar P^* of a point $P \in \mathcal{I} \subset \mathbb{R}^n$ is a hyperplane outside \mathcal{I} if and only if P is a *projective center* of \mathcal{I} , i.e. there is a projectivity ϖ such that $\varpi(P)$ is the affine center of $\varpi(\mathcal{I})$.

It is well known that a Hilbert geometry is the Cayley–Klein model of Bolyai's hyperbolic geometry if and only if it is given by an ellipsoid [2, 29.3].

A Hilbert geometry at a point O has *positive, non-negative, non-positive* and *negative curvature in the sense of Busemann* if there exists a neighborhood \mathcal{U} of O such that for every pair of points $P, Q \in \mathcal{U}$ we have

$$\begin{aligned} 2d_{\mathcal{I}}(\hat{P}, \hat{Q}) &> d_{\mathcal{I}}(P, Q), & 2d_{\mathcal{I}}(\hat{P}, \hat{Q}) &\geq d_{\mathcal{I}}(P, Q), \\ 2d_{\mathcal{I}}(\hat{P}, \hat{Q}) &\leq d_{\mathcal{I}}(P, Q), & 2d_{\mathcal{I}}(\hat{P}, \hat{Q}) &< d_{\mathcal{I}}(P, Q), \end{aligned}$$

respectively, where \hat{P}, \hat{Q} are the respective $d_{\mathcal{I}}$ -midpoints of the geodesic segments \overline{OP} and \overline{OQ} [3, (36.1) on p. 237]. If neither of the cases is satisfied in any neighborhood of O , then we say that the curvature is *indeterminate* [4, Definition 1]⁴. A projectivity ϖ is clearly a bijective isometry of $(\mathcal{I}, d_{\mathcal{I}})$ to $(\varpi(\mathcal{I}), d_{\varpi(\mathcal{I})})$, hence

(2.1) *Busemann's curvature is a projective invariant.*

3. PREPARATIONS

We consider a Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ and a point O in \mathcal{I} .

Lemma 3.1 ([4, Lemma 1 and Corollary]). *There exist two (maybe ideal) points X and Y in O^* such that line XY does not intersect \mathcal{I} , and $\partial\mathcal{I}$ is differentiable at the points in $\partial\mathcal{I} \cap (OX \cup OY)$.*

Proof. There is at least one chord \overline{AB} of \mathcal{I} which is bisected by O . Then the harmonic conjugate \bar{X} of O with respect to A and B is on the line at infinity.

⁴Notice that positivity or negativity of the curvature in [4, Definition 1] corresponds to non-negativity, respectively non-positivity in our terms.

If \bar{X} is the only point of O^* at infinity, then O^* cannot completely lie within the strip formed by the two supporting lines of \mathcal{I} which are parallel to AB , because otherwise, as O^* is a connected curve, it would intersect \mathcal{I} . Thus, a further point \bar{Y} of O^* outside this strip exists.

If \bar{X} is not the only point of O^* at infinity, then let that point be \bar{Y} .

Then line $\bar{X}\bar{Y}$ does not intersect \mathcal{I} , but intersects O^* in the points \bar{X} and \bar{Y} .

Since all but a denumerable set of points of $\partial\mathcal{I}$ are points of differentiability, we may choose points $X \in O^*$ and $Y \in O^*$ near \bar{X} and \bar{Y} , respectively, so that $\partial\mathcal{I}$ is differentiable at the points in $\partial\mathcal{I} \cap (OX \cup OY)$, and XY does not intersect \mathcal{I} . \square

Let ℓ_1 and ℓ_2 be straight lines through O , and let l_\pm be straight lines through O such that

$$(3.1) \quad -(\ell_1, \ell_2; l_-, l_+) \geq 1.$$

Denote by Y_\pm the points where l_\pm intersects $\partial\mathcal{I}$ so that

$$(3.2) \quad (Y_-, Y_+; O)^2 \leq 1.$$

Let t_\pm be the tangent lines of $\partial\mathcal{I}$ at Y_\pm .

Fix a coordinate system so that $O = (0, 0)$, l_- is the x -axis, l_+ is the y -axis, and Y_+ is in the upper half-plane. For x in a small neighborhood of 0, let y_\pm be the continuous functions of x such that $(x, y_\pm(x))$ are the two points of $\partial\mathcal{I}$ with abscissa x , and $Y_\pm = (0, y_\pm(0))$ so $\pm y_\pm(x) > 0$.

Fix an Euclidean metric d such that the two axes and the lines ℓ_1 and ℓ_2 are perpendicular to each other, respectively. Let $s > 0$ be the slope of ℓ_1 , hence the slope of ℓ_2 is $-1/s$. Let m_\pm be the slope of t_\pm , and if the intersection of t_\pm and the x -axis exists, then denote it by T_\pm . So Figure 3.1 shows what we have.

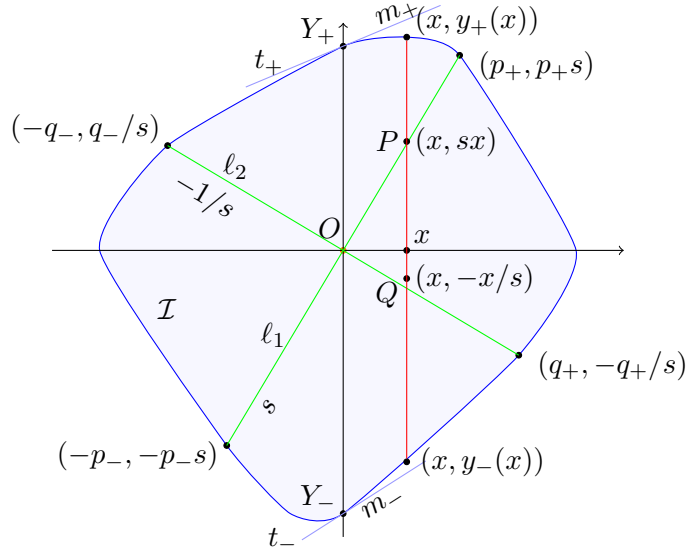


FIGURE 3.1. The configuration in euclidean plane.

Let $p_\pm, q_\pm > 0$ be such that $(\pm p_\pm, \pm p_\pm/s)$ are the points of $\ell_1 \cap \partial\mathcal{I}$, and $(\pm q_\pm, \mp q_\pm/s)$ are the points of $\ell_2 \cap \partial\mathcal{I}$.

Lemma 3.2. *If O is the affine midpoint of the chords $\ell_1 \cap \mathcal{I}$ and $\ell_2 \cap \mathcal{I}$, and the points Y_{\pm} are Alexandrov points of $\partial\mathcal{I}$, then in a small neighborhood of O we have*

$$(3.3) \quad d_{\mathcal{I}}(P, Q) - 2d_{\mathcal{I}}(\hat{P}, \hat{Q}) \geq \frac{x^2}{2s} \left(\frac{s^2 m_-}{y_-^2(0)} - \frac{m_+}{y_+^2(0)} \right) + x^3 O(1),$$

where \hat{P}, \hat{Q} are the $d_{\mathcal{I}}$ -midpoints of the geodesic segments \overline{OP} and \overline{OQ} , respectively.

Proof. Since O is the midpoint of the chords $\tilde{\ell}_1 = \ell_1 \cap \mathcal{I}$ and $\tilde{\ell}_2 = \ell_2 \cap \mathcal{I}$, we have $p := p_+ = p_-$ and $q := q_+ = q_-$.

We have to show that there is an $\varepsilon > 0$ such that the points $P = (x, sx) \in \tilde{\ell}_1$, $Q = (x, -x/s) \in \tilde{\ell}_2$ ($x \in (0, \varepsilon)$), and the respective $d_{\mathcal{I}}$ -midpoints \hat{P}, \hat{Q} of the geodesic segments \overline{OP} and \overline{OQ} satisfy (3.3).

The strict triangle inequality $d_{\mathcal{I}}(\hat{P}, \hat{Q}) < d_{\mathcal{I}}(\hat{P}, \bar{P}) + d_{\mathcal{I}}(\bar{P}, \bar{Q}) + d_{\mathcal{I}}(\bar{Q}, \hat{Q})$, where $\bar{P} = (\frac{x}{2}, \frac{sx}{2})$ and $\bar{Q} = (\frac{x}{2}, \frac{-x}{2s})$, gives

$$(3.4) \quad \begin{aligned} d_{\mathcal{I}}(P, Q) - 2d_{\mathcal{I}}(\hat{P}, \hat{Q}) \\ \geq (d_{\mathcal{I}}(P, Q) - 2d_{\mathcal{I}}(\bar{P}, \bar{Q})) - 2(d_{\mathcal{I}}(\hat{P}, \bar{P}) + d_{\mathcal{I}}(\bar{Q}, \hat{Q})), \end{aligned}$$

so it is enough to estimate the right-hand side of this inequality from below.

By (1.1) and the Taylor series expansion of the logarithm, we have

$$\begin{aligned} d_{\mathcal{I}}(O, P) &= \frac{1}{2} \ln \frac{p+x}{p-x} = \frac{1}{2} \left(\ln \left(1 + \frac{x}{p} \right) - \ln \left(1 - \frac{x}{p} \right) \right) \\ &= \frac{1}{2} \sum_{i=1}^{\infty} \frac{1 - (-1)^i}{i} \left(\frac{x}{p} \right)^i, \end{aligned}$$

hence

$$d_{\mathcal{I}}(O, P) = \sum_{j=0}^{\infty} \frac{1}{2j+1} \left(\frac{x}{p} \right)^{2j+1}, \quad \text{and} \quad d_{\mathcal{I}}(O, \bar{P}) = \sum_{j=0}^{\infty} \frac{2^{-1-2j}}{2j+1} \left(\frac{x}{p} \right)^{2j+1}.$$

The same calculation for Q and \bar{Q} leads to

$$d_{\mathcal{I}}(O, Q) = \sum_{j=0}^{\infty} \frac{1}{2j+1} \left(\frac{x}{q} \right)^{2j+1}, \quad \text{and} \quad d_{\mathcal{I}}(O, \bar{Q}) = \sum_{j=0}^{\infty} \frac{2^{-1-2j}}{2j+1} \left(\frac{x}{q} \right)^{2j+1}.$$

Further, as $d_{\mathcal{I}}(O, \hat{P}) = d_{\mathcal{I}}(O, P)/2$, and $d_{\mathcal{I}}(O, \hat{Q}) = d_{\mathcal{I}}(O, Q)/2$, the above formulas also imply

$$(3.5) \quad d_{\mathcal{I}}(\bar{P}, \hat{P}) = |d_{\mathcal{I}}(O, \hat{P}) - d_{\mathcal{I}}(O, \bar{P})| = \frac{1}{2} \sum_{j=1}^{\infty} \frac{1 - 2^{-2j}}{2j+1} \left(\frac{x}{p} \right)^{2j+1},$$

$$(3.6) \quad d_{\mathcal{I}}(\bar{Q}, \hat{Q}) = |d_{\mathcal{I}}(O, \hat{Q}) - d_{\mathcal{I}}(O, \bar{Q})| = \frac{1}{2} \sum_{j=1}^{\infty} \frac{1 - 2^{-2j}}{2j+1} \left(\frac{x}{q} \right)^{2j+1}.$$

Since $d_{\mathcal{I}}(P, Q) = \frac{1}{2} \left| \ln \left(\frac{sx - y_-(x)}{y_+(x) - sx} : \frac{-x/s - y_-(x)}{y_+(x) + x/s} \right) \right|$ by (1.1), the Taylor series expansion of the logarithm gives

$$\begin{aligned}
d_{\mathcal{I}}(P, Q) &= \frac{1}{2} \left(\ln \left(1 + \frac{sx}{-y_-(x)} \right) - \ln \left(1 - \frac{sx}{y_+(x)} \right) + \right. \\
&\quad \left. + \ln \left(1 + \frac{x/s}{y_+(x)} \right) - \ln \left(1 - \frac{x/s}{-y_-(x)} \right) \right) \\
&= \frac{1}{2} \left(\sum_{i=1}^{\infty} \frac{1 - (-1)^i}{i} \left(\frac{sx}{-y_-(x)} \right)^i + \sum_{i=1}^{\infty} \frac{1 - (-1)^i}{i} \left(\frac{x/s}{y_+(x)} \right)^i \right) \\
(3.7) \quad &= \sum_{j=0}^{\infty} \frac{-s^{2j+1}}{2j+1} \left(\frac{x}{y_-(x)} \right)^{2j+1} + \sum_{j=0}^{\infty} \frac{s^{-2j-1}}{2j+1} \left(\frac{x}{y_+(x)} \right)^{2j+1}.
\end{aligned}$$

In the same way $d_{\mathcal{I}}(\bar{P}, \bar{Q}) = \frac{1}{2} \left| \ln \left(\frac{sx/2 - y_-(x/2)}{y_+(x/2) - sx/2} : \frac{-x/2/s - y_-(x/2)}{y_+(x/2) + x/2/s} \right) \right|$ implies

$$(3.8) \quad d_{\mathcal{I}}(\bar{P}, \bar{Q}) = \sum_{j=0}^{\infty} \frac{-s^{2j+1}}{2j+1} \left(\frac{x/2}{y_-(x/2)} \right)^{2j+1} + \sum_{j=0}^{\infty} \frac{s^{-2j-1}}{2j+1} \left(\frac{x/2}{y_+(x/2)} \right)^{2j+1}.$$

Since the points Y_{\pm} are Alexandrov points of $\partial \mathcal{I}$, we have the Taylor series expansions $\bar{y}_{\pm}(t) = \bar{y}_{\pm}(0) + t\bar{y}'_{\pm}(0) + t^2O(1)$ of the functions $\bar{y}_{\pm} := 1/y_{\pm}$. For easy handling of this we define $\bar{y}_{\pm}^{(i)}(0)$ ($i = 0, 1, 2$) so that $\bar{y}_{\pm}(t) = \sum_{i=0}^2 t^i \bar{y}_{\pm}^{(i)}(0)/i!$.

Substituting (3.5), (3.6), (3.7), (3.8), and the above Taylor expansion of $\bar{y}_{\pm}(x)$ into the right-hand side of (3.4), we obtain

$$\begin{aligned}
&(d_{\mathcal{I}}(P, Q) - 2d_{\mathcal{I}}(\bar{P}, \bar{Q})) - 2(d_{\mathcal{I}}(\hat{P}, \bar{P}) + d_{\mathcal{I}}(\bar{Q}, \hat{Q})) \\
&= \sum_{j=0}^{\infty} \frac{-s^{2j+1}}{2j+1} \left(\sum_{i=0}^2 x^{i+1} \frac{\bar{y}_{-}^{(i)}(0)}{i!} \right)^{2j+1} + \sum_{j=0}^{\infty} \frac{s^{-2j-1}}{2j+1} \left(\sum_{i=0}^2 x^{i+1} \frac{\bar{y}_{+}^{(i)}(0)}{i!} \right)^{2j+1} - \\
&\quad - 2 \sum_{j=0}^{\infty} \frac{-s^{2j+1}}{2j+1} \left(\sum_{i=0}^2 x^{i+1} \frac{\bar{y}_{-}^{(i)}(0)}{i!2^{i+1}} \right)^{2j+1} - 2 \sum_{j=0}^{\infty} \frac{s^{-2j-1}}{2j+1} \left(\sum_{i=0}^2 x^{i+1} \frac{\bar{y}_{+}^{(i)}(0)}{i!2^{i+1}} \right)^{2j+1} - \\
&\quad - \sum_{j=1}^{\infty} \frac{1 - 2^{-2j}}{2j+1} \left(\frac{x}{p} \right)^{2j+1} - \sum_{j=1}^{\infty} \frac{1 - 2^{-2j}}{2j+1} \left(\frac{x}{q} \right)^{2j+1}.
\end{aligned}$$

Separating the summands with index $j = 0$ from the sums with running variable j , and moving them to the beginning result in

$$\begin{aligned}
&(d_{\mathcal{I}}(P, Q) - 2d_{\mathcal{I}}(\bar{P}, \bar{Q})) - 2(d_{\mathcal{I}}(\hat{P}, \bar{P}) + d_{\mathcal{I}}(\bar{Q}, \hat{Q})) \\
&= -s \sum_{i=0}^2 x^{i+1} \frac{\bar{y}_{-}^{(i)}(0)}{i!} + \frac{1}{s} \sum_{i=0}^2 x^{i+1} \frac{\bar{y}_{+}^{(i)}(0)}{i!} + s \sum_{i=0}^2 x^{i+1} \frac{\bar{y}_{-}^{(i)}(0)}{i!2^i} - \\
&\quad - \frac{1}{s} \sum_{i=0}^2 x^{i+1} \frac{\bar{y}_{+}^{(i)}(0)}{i!2^i} + x^3 O(1).
\end{aligned}$$

The summands with index $i = 0$ just cancel each other, the summands with index $i = 2$ has multiplier x^3 , so we obtain

$$\begin{aligned}
&(d_{\mathcal{I}}(P, Q) - 2d_{\mathcal{I}}(\bar{P}, \bar{Q})) - 2(d_{\mathcal{I}}(\hat{P}, \bar{P}) + d_{\mathcal{I}}(\bar{Q}, \hat{Q})) \\
&= x^2 \left(\frac{1}{2s} \bar{y}'_{+}(0) - \frac{s}{2} \bar{y}'_{-}(0) \right) + x^3 O(1).
\end{aligned}$$

Since $y_{\pm} := 1/\bar{y}_{\pm}$, one gets

$$e := \frac{1}{2s}\bar{y}'_+(0) - \frac{s}{2}\bar{y}'_-(0) = \frac{-1}{2s}\frac{y'_+(0)}{y_+^2(0)} + \frac{s}{2}\frac{y'_-(0)}{y_-^2(0)} = \frac{1}{2s}\left(\frac{s^2 m_-}{y_-^2(0)} - \frac{m_+}{y_+^2(0)}\right)$$

that proves the lemma. \square

4. CURVATURE IN HILBERT GEOMETRY

Firstly we reprove the result of [6] using our preparatory Lemma 3.2.

Theorem 4.1. *A Hilbert geometry can not have positive or non-negative curvature at any point.*

Proof. It is enough to prove that

(4.1) *through every point O of a Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ there are two geodesics $\tilde{\ell}_1$ and $\tilde{\ell}_2$ such that in any suitable small open neighborhood \mathcal{U} of O inequality $2d_{\mathcal{I}}(\hat{P}, \hat{Q}) < d_{\mathcal{I}}(P, Q)$ is fulfilled for some points $P \in \tilde{\ell}_1 \cap \mathcal{U}$ and $Q \in \tilde{\ell}_2 \cap \mathcal{U}$, where $\hat{P}, \hat{Q} \in \mathcal{U}$ are the $d_{\mathcal{I}}$ -midpoints of the geodesic segments \overline{OP} and \overline{OQ} , respectively.*

As two geodesics lie always in a common plane, it is enough to prove (4.1) in the plane. Let O be an arbitrary point in $\mathcal{I} \subset \mathbb{R}^2$.

By Lemma 3.1, there is a projectivity ϖ such that $\varpi(O)$ is the affine center of at least two geodesics $\varpi(\tilde{\ell}_1)$ and $\varpi(\tilde{\ell}_2)$. So taking (2.1) into account, we assume from now on that O is the affine center of the segments $\ell_1 \cap \mathcal{I}$ and $\ell_2 \cap \mathcal{I}$.

Choose the straight lines l_{\pm} through O so that Y_{\pm} are Alexander points of $\partial\mathcal{I}$, and $-(\ell_1, \ell_2; l_-, l_+) > 1$. This is possible because if equality happened in (3.1), then rotating l_- a little bit helps. So by (3.2) we have

$$(4.2) \quad -(\ell_1, \ell_2; l_-, l_+) > (Y_-, Y_+; O)^2.$$

If either one of the tangents t_{\pm} is parallel to l_- , then slightly rotate l_- around O so that it keeps the properties required above and intersects the tangents t_{\pm} in some points, say $T_{\pm} = t_{\pm} \cap l_-$. If $|(T_+, T_-; O)| < |(Y_+, Y_-; O)|$, then change the indexing from \pm to \mp , so we have $|(T_+, T_-; O)| \geq |(Y_+, Y_-; O)|$.

Now we choose a coordinate system so that the positive half of the x -axis contains T_- . Figure 4.1 shows what we have if $O \in \mathcal{I} \cap \overline{T_+ T_-}$.

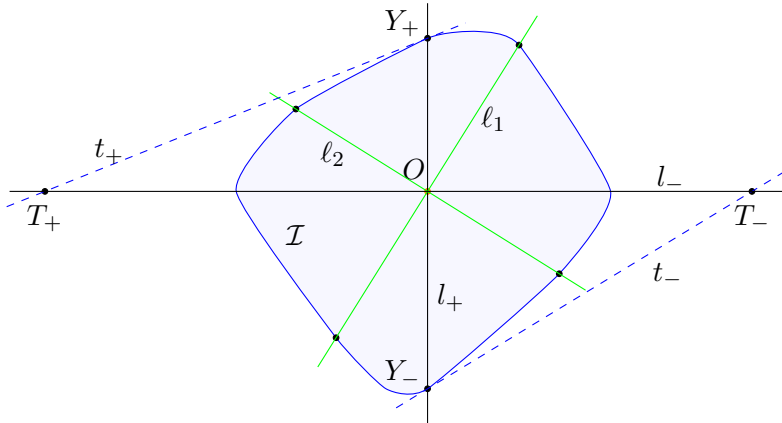


FIGURE 4.1. The affine configuration if $O \in \mathcal{I} \cap \overline{T_+ T_-}$.

By Lemma 3.2 statement (4.1) fulfills if $\frac{m_-}{2sy_+^2(0)}(s^2\frac{y_+^2(0)}{y_-^2(0)} - \frac{m_+}{m_-})$, the main term in (3.3), is positive, i.e. $s^2\frac{y_+^2(0)}{y_-^2(0)} > \frac{m_+}{m_-}$. Observe that (4.2) implies

$$s^2\frac{y_+^2(0)}{y_-^2(0)} = -\frac{-s|Y_+O|^2}{1/s|OY_-|^2} = \frac{-(\ell_1, \ell_2; l_-)}{(Y_-, Y_+; O)^2} = \frac{-(\ell_1, \ell_2; l_-, l_+)}{(Y_-, Y_+; O)^2} > 1.$$

So we need to prove that $\frac{m_+}{m_-} \leq 1$. If $0 < (T_+, T_-; O)$, then $m_+ < 0$ and therefore $\frac{m_+}{m_-} < 0$. If $(T_+, T_-; O) < 0$, then

$$\frac{m_+}{m_-} = \frac{|Y_+O|/|T_+O|}{|OY_-|/|OT_-|} = \frac{|Y_+O|}{|OY_-|} \frac{|OT_-|}{|T_+O|} = \frac{|(Y_+, Y_-; O)|}{|(T_+, T_-; O)|} \leq 1,$$

so the proof is complete. \square

We use again Lemma 3.2 to improve [4, the first statement of Theorem].

Theorem 4.2. *A point O in the Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ has non-positive curvature if and only if it is a projective center of \mathcal{I} .*

Proof. Firstly we prove the necessity part⁵.

We assume that $(\mathcal{I}, d_{\mathcal{I}})$ has non-positive curvature at O , and have to prove that O^* is a hyperplane. For this it is enough to prove that every plane section of O^* is a straight line. So, from now on we assume that $\mathcal{I} \subset \mathbb{R}^2$, and need to prove that

there is a projectivity ϖ such that $\varpi(O)$ is the affine center of $\varpi(\mathcal{I})$.

By Lemma 3.1, there is a projectivity ϖ such that $\varpi(O)$ is the affine center of at least two geodesics $\varpi(\tilde{\ell}_1)$ and $\varpi(\tilde{\ell}_2)$, so, according to (2.1), we may assume without loss of generality that O is the affine center of the segments $\ell_1 \cap \mathcal{I}$ and $\ell_2 \cap \mathcal{I}$.

This time we choose the straight lines l_{\pm} through O so that

$$(4.3) \quad -(\ell_1, \ell_2; l_-, l_+) = 1,$$

Y_{\pm} are Alexander points of $\partial\mathcal{I}$, and l_- intersects both t_{\pm} . This can be achieved easily, because except the two directions, where l_- is parallel to one of the tangents t_{\pm} , and where a point Y_{\pm} is not an Alexander point of $\partial\mathcal{I}$, the direction of l_- can be chosen freely, and l_+ is determined change accordingly by (4.3).

Choose the direction of the x -axes so that the abscissa of T_- be positive. Again Figure 4.1 shows what we have if $O \in \overline{T_+T_-}$.

Since the Busemann curvature is non-positive, i.e. $2d_{\mathcal{I}}(\hat{P}, \hat{Q}) \geq d_{\mathcal{I}}(P, Q)$, the main term in (3.3) of Lemma 3.2 should vanish, i.e. $\frac{s^2m_-}{y_-^2(0)} = \frac{m_+}{y_+^2(0)}$. However $s^2 = -\frac{-s}{1/s} = -(\ell_1, \ell_2; l_-) = -(\ell_1, \ell_2; l_-, l_+) = 1$, so $-\frac{y'_-(0\pm)}{y_-^2(0)} = \frac{y'_+(0\pm)}{y_+^2(0)}$ follows, where the sign \pm at $0\pm$ is determined by the direction of the x -axis. Rearrangement gives

$$y_+(0)\frac{y_+(0)}{y'_+(0\pm)} = (-y_-)(0)\frac{(-y_-)(0)}{(-y_-)'(0\pm)},$$

⁵This is [4, first statement of Theorem]

that, as $\pm y_{\pm}(0) = d(O, Y_{\pm})$ and $\pm y_{\pm}(0)/(\pm y'_{\pm}(0)) = d(O, T_{\pm})$, means that the triangles $\triangle OY_+T_+$ and $\triangle OY_-T_-$ have equal areas.

Change now to a Euclidean metric d_e such that ℓ_1 and ℓ_2 are orthogonal. Let the direction vector of l_+ be $(\cos \varphi, \sin \varphi)$, hence the direction vector of l_- is $(\cos \varphi, -\sin \varphi)$, and let r be the radial function of $\partial \mathcal{I}$ from the point O , hence $Y_+ = r(\varphi)(\cos \varphi, \sin \varphi)$ and $Y_- = r(\varphi + \pi)(\cos(\varphi + \pi), \sin(\varphi + \pi))$. See Figure 4.2.

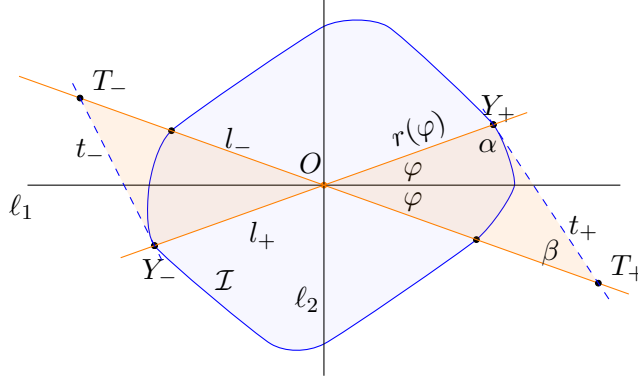


FIGURE 4.2. We have $\text{area}(\triangle OY_+T_+) = \text{area}(\triangle OY_-T_-)$ for every φ .

Define $\alpha := \angle(O, Y_+, T_+)$, $\beta := \pi - \alpha - 2\varphi$. Then $\cot \alpha = -\dot{r}(\varphi)/r(\varphi)$ and $a(\varphi) := 2 \text{area}(\triangle OY_+T_+) = r^2(\varphi) \frac{\sin(2\varphi)}{\sin \beta} \sin \alpha$, hence

$$\begin{aligned} \frac{\sin(2\varphi)}{a(\varphi)} &= r^{-2}(\varphi) \frac{\sin(2\varphi + \alpha)}{\sin \alpha} = r^{-2}(\varphi) (\sin(2\varphi) \cot \alpha + \cos(2\varphi)) \\ &= \sin(2\varphi) \frac{-\dot{r}(\varphi)}{r^3(\varphi)} + \cos(2\varphi) \frac{1}{r^2(\varphi)} = \frac{1}{2} \left(\frac{\sin(2\varphi)}{r^2(\varphi)} \right)'. \end{aligned}$$

Thus, we have

$$\left(\frac{\sin(2\varphi)}{r^2(\varphi)} \right)' = \frac{\sin(2\varphi)}{a(\varphi)} = \frac{\sin(2(\varphi + \pi))}{a(\varphi + \pi)} = \left(\frac{\sin(2(\varphi + \pi))}{r^2(\varphi + \pi)} \right)',$$

and also $\lim_{\varphi \rightarrow 0} \frac{\sin(2\varphi)}{r^2(\varphi)} = 0 = \lim_{\varphi \rightarrow 0} \frac{\sin(2(\varphi + \pi))}{r^2(2(\varphi + \pi))}$. Thus $r(\varphi) \equiv r(\varphi + \pi)$ follows, meaning that \mathcal{I} is affine symmetric with respect to O .

Thus the necessity part of the theorem is proved.

Next we prove the sufficiency part⁶.

We assume that O is a projective center of \mathcal{I} , and we have to prove that

(4.4) *there is a suitable small open neighborhood \mathcal{U} of O that for every geodesics $\tilde{\ell}_1$ and $\tilde{\ell}_2$ through O inequality $2d_{\mathcal{I}}(\hat{P}, \hat{Q}) \leq d_{\mathcal{I}}(P, Q)$ is fulfilled for every points $P \in \tilde{\ell}_1 \cap \mathcal{U}$ and $Q \in \tilde{\ell}_2 \cap \mathcal{U}$, where $\hat{P}, \hat{Q} \in \mathcal{U}$ are the $d_{\mathcal{I}}$ -midpoints of the geodesic segments \overline{OP} and \overline{OQ} , respectively.*

According to (2.1), we may assume without loss of generality that O is the affine center of \mathcal{I} . Since two geodesics lie in a common plane, it is enough to prove (4.4) in the plane, so we assume that O is the affine center of $\mathcal{I} \subset \mathbb{R}^2$.

Choose the straight lines l_{\pm} so that Y_{\pm} are Alexander points of $\partial \mathcal{I}$, and

$$(4.5) \quad -(\ell_1, \ell_2; l_-, l_+) > 1.$$

⁶The last paragraph of [6] argues that this “does not seem easy”.

This is possible because if equality happened in (3.1), then rotating l_- a little bit helps. Moreover, if t_+ is parallel to l_- , then one can slightly rotate l_- around O so that (4.5) remains valid and intersects t_+ . Thus, we can assume that the point T_+ exists. Since O is the affine center of \mathcal{I} , we have $t_+ \parallel t_-$, so also point T_- exists, and O is clearly the affine center of $\overline{T_-T_+}$.

Now we fix the coordinate system and euclidean metric given in Section 3 so that the positive half of the x -axes contains T_- . Again Figure 4.1 shows what we have.

By Lemma 3.2 statement (4.4) fulfills if the main term $\frac{m_-}{2sy_+^2(0)}(s^2\frac{y_+^2(0)}{y_-^2(0)} - \frac{m_+}{m_-})$ in (3.3) is positive. This fulfills because $\frac{m_+}{m_-} = 1$ by $t_+ \parallel t_-$, $m_- > 0$, and $s^2\frac{y_+^2(0)}{y_-^2(0)} - 1 = -\frac{-s}{1/s} - 1 = -(\ell_1, \ell_2; l_-, l_+) - 1 > 0$ by (4.5). \square

5. CONSEQUENCES

The following statements sharpen and extend the solution [4, second statement in Theorem] of Kelly and Strauss given to Busemann's [3, Problem 34, p. 406].

Theorem 5.1. *A Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic geometry if and only if there is a hyperplane intersecting the Hilbert geometry so that every point of the intersection is of non-positive curvature.*

Proof. If the Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic geometry, then it has non-positive curvature at every point.

If there is a hyperplane intersecting the Hilbert geometry so that the Hilbert geometry has non-positive curvature at every point in the intersection, then all these points are projective centers by Theorem 4.2, and therefore [7, Theorem 3.3(a)] implies that the domain is an ellipsoid, hence the Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic geometry. \square

For dimension 2 we have an even sharper version.

Theorem 5.2. *A 2-dimensional Hilbert geometry is a Cayley–Klein model of the hyperbolic space if and only if it has two points of non-positive curvature and its boundary is twice differentiable where it is intersected by the line joining those points of non-positive curvature.*

Proof. If the 2-dimensional Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic plane, then it has non-positive curvature at every point.

If the 2-dimensional Hilbert geometry has two points of non-positive curvature and its boundary is twice differentiable where it is intersected by the line joining those points of non-positive curvature, then [5, Theorem 3] implies that the domain is an ellipse. \square

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BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1, 6725 SZEGED, HUNGARY
E-mail: kurusa@math.u-szeged.hu .